PHASE MARGIN REVISITED: PHASE-ROOT LOCUS, BODE PLOTS, AND PHASE SHIFTERS

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Supplementary materials

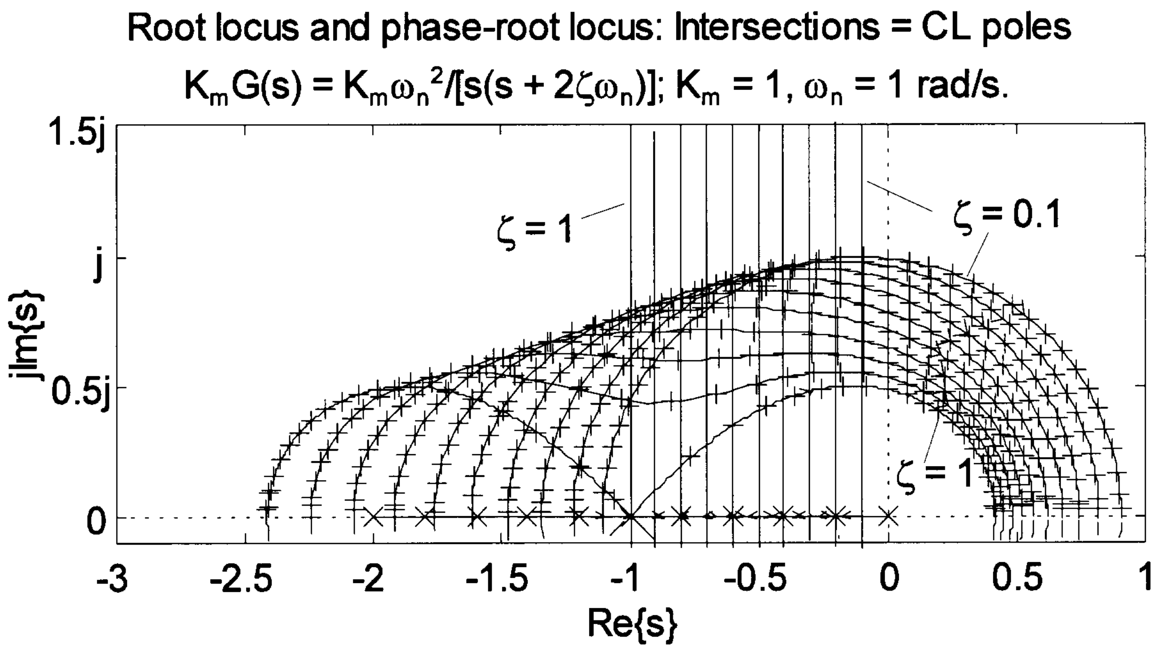
1: TRANSLATION OF RL STABILITY CRITERION INTO BODE GAIN-BASED CRITERION FOR MINIMUM-PHASE SYSTEMS.

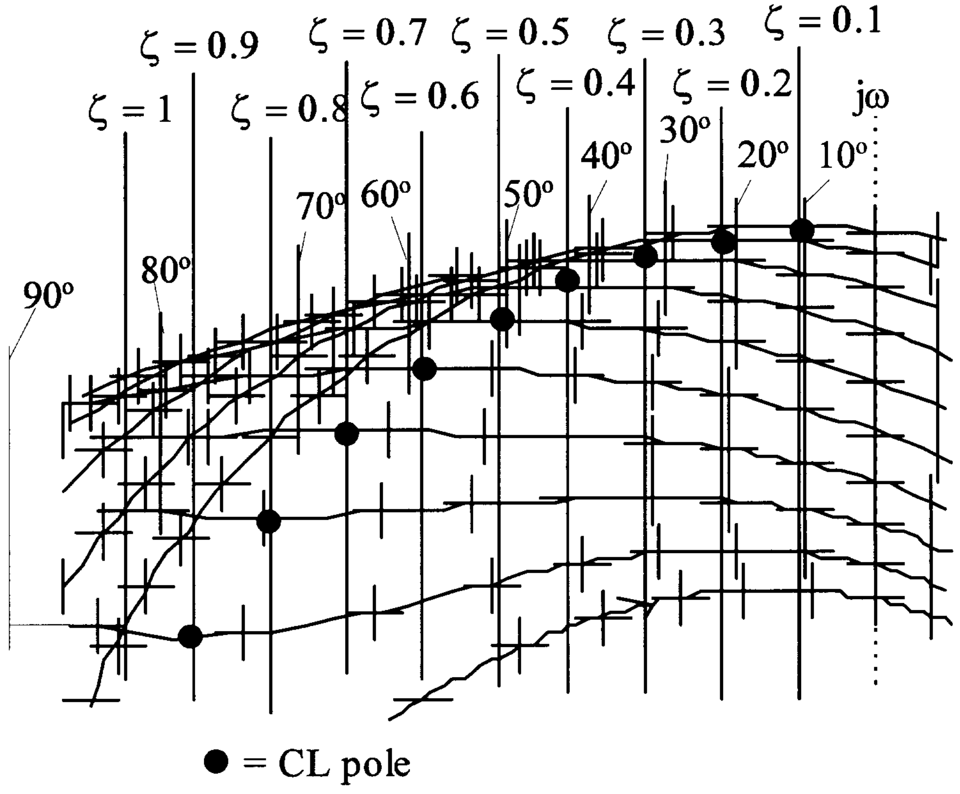
For minimum-phase systems, GM and PM have their usual meanings; e.g., an increase in gain is typically required to move the closed-loop pole into the right half-plane. [Note, however, that there are rare exceptions, such as $G(s) = (s^2 + 2s + 401)/[s^2(s + 100)(s^2 + 20s + 1.01 \cdot 10^4)$, for which the closed-loop system is unstable for $K_m < 1.38 \cdot 10^6$.] The $j\omega$-axis is the boundary between stability and instability. Thus for such systems, if $s_i(K_m)$ is imaginary for $i = i_1$ and $K_m = K_0$, then the system is stable for $K_m < K_0$ and unstable for $K_m > K_0$. [It is assumed that K_0 is the lowest value of K_m for which there is an imaginary $s_i(K_m)$.] Because $s_{i1}(K_0)$ is imaginary, we may write $s_{i1}(K_0) = j\omega_{pc}$ because, as $s_{i1}(K_0)$ is on the root locus, $\angle K_0G(s_{i1}(K_0)) = \angle K_0G(j\omega_{pc}) = -180°$ so that $\angle K_mG(j\omega)$ crosses over from greater than -180° to less than -180° at $\omega = \omega_{pc}$. Because the magnitude condition is satisfied for the gain-modified plant at $s = s_{i1}(K_0) = j\omega_{pc}$, it follows that $K_0 = 1/|G(j\omega_{pc})|$. Thus for stability, $K_m < 1/|G(j\omega_{pc})|$, or $|K_mG(j\omega_{pc})| < 1$, which is the required gain stability condition on the frequency response $K_mG(j\omega)$. The factor by which $K_m < K_0$ is called the GM, namely $GM(K_m) = K_0/K_m = 1/|K_mG(j\omega_{pc})|$ or in dB, $GM_{dB}(K_m) = -20 \cdot \log_{10}\{|K_mG(j\omega_{pc})|\}$.

2: RESONANCES DO NOT ALWAYS SIGNIFICANTLY AFFECT STABILITY MARGINS. Although it was true in the example in the main paper that the resonance directly determined the stability margins, this is not always the case. If the resonance occurs at a frequency well below the crossover frequencies, the resonance may have practically no effect. For example, for $K_mG(s) = K_m \bullet (s + 10)(s + 1)/[s(s + 1000)(s + 800)(s + 400)(s^2 + 2\zeta\omega_n s + \omega_n^2)$ where $K_m = 1 \bullet 10^9$, the PMs for $\zeta = 0.2$ and $2 \bullet 10^{-6}$ (i.e., a very weak peak to a very sharp one) are, respectively, 42.89° and 42.4°, and the respective GMs are 10.14 dB and 10.07 dB. Even if ζ is increased to 0.7, the PM is 44.6° and the GM is 10.32 dB. From the view of PRL, the gain is sufficiently large so that there are not individual PRLs around the open-loop poles, but rather only one large PRL around the entire open-loop pole structure. The large PRL, way outside the open-loop poles, is unaffected by the details of any of the individual open-loop poles, including the resonant pole near the $j\omega$ axis. The PRL is far from the poles when K_m is large because $|K_mG(s)| = 1$ on the PRL, which requires $|G(s)|$ to be very small and thus s to be very far from the poles of G(s) for s to be on the PRL.

3: VISUALIZATION OF THE VALIDITY OF PM = 100ζ FOR CANONICAL SECOND-ORDER SYSTEMS USING PHASE-ROOT LOCUS.
Without loss of generality, here we assume that ω_n = 1 rad/s.
In Fig. a below (next page) are shown the overlayed RL and calibrated PRL for $0 < \zeta \le 1$ in steps of 0.1. The breakaway values of s on the RL are $s_{b\ell} = -\zeta_\ell$. It is remarkable but predictable that even though the phase-root loci are noncircular (except for $\zeta = 0$), all their intersections with the associated root loci and thus the closed-loop poles (shown as dots) fall on a circle [namely, the $\zeta = 0$ PRL, which for $G(s) = \omega_n^2/s^2 = 1/s^2$ is the unit circle]. In this simple case, the closed-loop poles are at (ω_n = 1 rad/s) $\angle \pm(\pi - \cos^{-1}(\zeta))$.

The + signs in the figure below (next page) are the PRL calibration marks for every 10°; each represents the PM for a closed-loop pole at that point along the PRL. For the upper half-plane, one may multiply $K_mG(s)$ (holding $K_m = 1$ and ζ fixed) by $e^{j\theta}$ for various values of θ, and observe the resulting root loci and their intersections with the PRL (the closed-loop poles), which for $\theta = 10° \bullet \ell$ - PM(1) are the + signs in Fig. 4. As phase is added, the RL swings around, and the closed-loop pole location changes direcfly. b below zooms in to show the PM = 100ζ locations; they are seen to fall quite close to the closed-loop poles for most ζ up to 0.7, thus validating the approximate relation PM = 100ζ for the canonical second-order system. Note that the radius of each closed-loop pole is ω_n = 1 rad/s and the ω value is $\omega_d = \omega_n\{1 - \zeta^2\}^{1/2}$. The intersection of each PRL with the $j\omega$ axis is ω_{gc} for that value of ζ. Enforcement of the magnitude condition and geometrical analysis show that the values of s on the PRL for the canonical second-order system satisfy $s = \alpha\omega_n \angle f$, where f ranges from 0 to 2π and α satisfies the quartic equation $\alpha^4 + 4\zeta\cos(f)\alpha^3 + 4\zeta^2\alpha^2 = 1$. The special cases $f = \pi - \cos^{-1}(\zeta)$ and f = 90° substituted into this quartic can easily be analyzed to give, respectively, $|s| = \omega_n$ (closed-loop pole for K_m = 1) and $|s| = \omega_{gc} = \omega_n\{[4\zeta^4 + 1]^{1/2} - 2\zeta^2\}^{1/2}$ ($j\omega$ -axis crossing of PRL).


Root locus and phase-root locus: Intersections = CL poles
$K_mG(s) = K_m\omega_n^2/[s(s + 2\zeta\omega_n)]$; $K_m = 1$, $\omega_n = 1$ rad/s.
1.5j
j
0.5j
0
jIm{s}
ζ = 1
ζ = 0.1
ζ = 1
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
Re{s}


ζ = 0.9
ζ = 0.7
ζ = 0.5
ζ = 0.3
ζ = 0.1
ζ = 1
ζ = 0.8
ζ = 0.6
ζ = 0.4
ζ = 0.2
jω
30°
40°
20°
10°
70°
60°
50°
80°
90°
● = CL pole

4: The effects of a UCSPS on real-axis closed-loop poles.

If a closed-loop pole is on a real-axis RL branch, it and its associated RL branch do not move under a negative UCSPS if the closed-loop pole is to the right of its RL-originating open-loop pole, nor under a positive UCSPS if the closed-loop pole is to the left of its RL-originating open-loop pole, as the resulting phase lies between the +180° and -180° discontinuity that occurs on the real axis from one side of the RL branch to the other side. If, for example, a negative UCSPS is added and the real closed-loop pole is to the left of the open-loop pole, no problem exists: Just add the negative phase to the greater-than-(-180°) phases of KG(s) in the upper half-plane and the corresponding positive phase to the less-than-(+180°) phases of KG(s) in the lower half-plane. Note that for plants with more poles than zeros, the phase above the real axis just to the right of the rightmost pole is always negative. Between real open-loop poles with intervening RL branches exist either real zeros or breakaway branches (phase passes, respectively, through 0° or a +/-180° discontinuity). These latter facts justify the preceding comments.

5: General expressions for $g_{ps}(t)$, the impulse response of the UCSP-shifted system.

It is interesting to note that if g(t) were individual bicausal sinusoids [$\sigma_\ell = 0$ and $-\infty < t < \infty$ in (3a) of the main paper], then (3c) with $\sigma_\ell = 0$ **would** apply, because the FT terms would be totally localized to ω_ℓ (and thus would not extend into the other half-plane). The sampling property of the Dirac delta function would then be used to evaluate the LT^{-1}, not the residue theorem. In general, all one can write exactly are the following integrals, which cannot be evaluated in closed form:

$$g_{ps}(t) = \sum_{\ell=1}^{N} A_\ell \int_{-\infty}^{\infty} \frac{\sigma_\ell \cos(\varphi_\ell) - \omega_\ell \sin(\varphi_\ell) + j\omega \bullet \cos(\varphi_\ell)}{\omega_\ell^2 + \sigma_\ell^2 - \omega^2 + j2\sigma_\ell \omega} e^{j\omega(t+\theta/|\omega|)} \, d\omega/(2\pi), \quad \text{all } t \qquad \text{(W1a)}$$

$$= \frac{1}{\pi} \int_0^{\infty} |G(j\omega)| \bullet \cos\{\omega t + \angle G(j\omega) + \theta\} \, d\omega, \quad \text{all } t, \qquad \text{(W1b)}$$

where (W1a) [in which use is made of the unilateral FT of $e^{-\sigma_\ell t} \cos(\omega_\ell t + \varphi_\ell)$] is true only for g(t) in (3a), whereas (W1b) holds for any real g(t).